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Construction of an atomic decomposition for functions with compact support

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Abstract

Chang, Krantz and Stein [D.-C. Chang, S.G. Krantz, E.M. Stein, H^p theory on a smooth domain in $\mathbb{R}^n$ and elliptic boundary value problems, J. Funct. Anal. 114 (1993) 286–347] proved that if $f \in H^p(\mathbb{R}^n)$ and f vanishes outside $\overline{\Omega}$, then f has an atomic decomposition whose atoms are contained in Ω. The purpose of this paper is to give another proof for the case $n/(n+1) < p \leqslant 1$ and Ω a cube. Our argument provides a simple, direct construction of the desired atomic decomposition, and it works in a class of function spaces more general than the usual Hardy spaces.

Keywords: Hardy space; Atomic decomposition; Compact support

1. Introduction

Suppose $f \in H^p(\mathbb{R}^n)$ and f vanishes outside $\overline{\Omega}$. Then does f have an atomic decomposition whose atoms are contained in Ω? In [1, Theorem 3.3] Chang, Krantz, and Stein used an argument based on the square function to prove that the answer is "yes" for a special Lipschitz domain.

The purpose of this paper is to give a more direct proof for the case $n/(n+1) < p \leqslant 1$ and Ω a cube. Our argument works more generally for the spaces $H_U^{\Phi,q}(\mathbb{R}^n)$, which we define in Section 2. We state our result (Theorem 3) at the end of Section 2. Section 3 is for the preliminaries. We prove our result in Section 4. For the background of the basic idea, see also [3,5].

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2. Definition of $H_U^{\Phi,q}(\mathbb{R}^n)$ and main results

A function $\theta:(0,+\infty)\to(0,+\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C>0$ such that

$$\theta(r)\leqslant C\theta(s)\quad\bigl(\theta(r)\geqslant C\theta(s)\bigr)\quad\text{for } r\leqslant s.$$

A function $\theta:(0,+\infty)\to(0,+\infty)$ is said to satisfy the doubling condition if there exists a constant $C>0$ such that

$$C^{-1}\leqslant\frac{\theta(r)}{\theta(s)}\leqslant C\quad\text{for } \frac{1}{2}\leqslant\frac{r}{s}\leqslant 2.$$

Let $\mathcal{F}$ be the set of all continuous, increasing and bijective functions $\Phi:[0,+\infty)\to[0,+\infty)$. Then $\Phi(0)=0$ and $\lim_{r\to+\infty}\Phi(r)=+\infty$ for $\Phi\in\mathcal{F}$. Let $\mathfrak{D}'$ be the space of distributions on $\mathbb{R}^n$.

Definition 1. Let $\Phi\in\mathcal{F}$, $1<q\leqslant\infty$ and $r^{1/q}\Phi^{-1}(1/r)$ be almost decreasing. A function a on $\mathbb{R}^n$ is called a (Φ,q)-atom if there exists a cube Q with sides parallel to the axes such that

$$\begin{cases}\text{(i)} & \operatorname{supp} a\subset\overline{Q},\\ \text{(ii)} & \|a\|_q\leqslant|Q|^{1/q}\Phi^{-1}\bigl(\frac{1}{|Q|}\bigr),\\ \text{(iii)} & \int a(x)\,dx=0,\end{cases}$$

where $\|a\|_q$ is the L^q norm of a, $\overline{Q}$ is the closure of Q and $|Q|$ is the Lebesgue measure of Q. We denote by $A(\Phi,q)$ the set of all (Φ,q)-atoms.

Definition 2. Let $\Phi\in\mathcal{F}$, $1<q\leqslant\infty$, and let $r^{1/q}\Phi^{-1}(1/r)$ be almost decreasing, $U\in\mathcal{F}$ and U be concave. We define the space $H_U^{\Phi,q}(\mathbb{R}^n)\subset\mathfrak{D}'$ as follows:

$f\in H_U^{\Phi,q}(\mathbb{R}^n)$ if and only if there exist sequences $\{a_j\}\subset A(\Phi,q)$ and positive numbers $\{\lambda_j\}$ such that

$$f=\sum_j\lambda_j a_j\quad\text{in }\mathfrak{D}'\qquad\text{and}\qquad\sum_j U(\lambda_j)<+\infty.\tag{1}$$

In general, the expression (1) is not unique. We define

$$\|f\|_{H_U^{\Phi,q}}=\inf\biggl\{U^{-1}\biggl(\sum_j U(\lambda_j)\biggr):\ f=\sum_j\lambda_j a_j\ \text{in }\mathfrak{D}'\biggr\},$$

where the infimum is taken over all expressions (1).

$H_U^{\Phi,q}(\mathbb{R}^n)$ is a linear space. Let $d(f,g)=U(\|f-g\|_{H_U^{\Phi,q}})$ for $f,g\in H_U^{\Phi,q}(\mathbb{R}^n)$. Then $d(f,g)$ is a metric and $H_U^{\Phi,q}(\mathbb{R}^n)$ is complete with respect to this metric. Let $I(r)=r$. Then $\|f\|_{H_I^{\Phi,q}}$ is a norm and $H_I^{\Phi,q}$ is a Banach space. If $\Phi(r)=U(r)=r^p$, $n/(n+1)<p\leqslant 1$ and $1<q\leqslant\infty$, then $H_U^{\Phi,q}(\mathbb{R}^n)$ is the usual $H^p(\mathbb{R}^n)$. Let

$$L^{q,0}_{\mathrm{comp}}\bigl(\mathbb{R}^n\bigr)=\biggl\{f\in L^q_{\mathrm{comp}}\bigl(\mathbb{R}^n\bigr):\ \int f(x)\,dx=0\biggr\}.$$

Then $L^{q,0}_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $H_U^{\Phi,q}(\mathbb{R}^n)$.

Theorem 3. *Let Φ, q and U satisfy the assumptions of Definition* 2. *Let $r\Phi^{-1}(1/r)$ be almost decreasing and $r^{(n+1)/n}\Phi^{-1}(1/r)$ be almost increasing. If $f \in L^{q,0}_{\mathrm{comp}}(\mathbb{R}^n)$ and $\operatorname{supp} f \subset \overline{Q}$ for some cube Q, then there exist sequences $\{a_j\} \subset A(\Phi, q)$ and positive numbers $\{\lambda_j\}$ such that*

$$f = \sum_j \lambda_j a_j \quad \text{in } \mathfrak{D}', \qquad \bigcup_j \operatorname{supp} a_j \subset \overline{Q}, \qquad \sum_j U(\lambda_j) \leqslant CU\bigl(\|f\|_{H^{\Phi,q}_U}\bigr),$$

where $C > 0$ is dependent only on n, Φ and U.

3. Preliminaries

To prove Theorem 3, we define generalized Campanato spaces and state a lemma.

For $z = (z^{(1)}, \ldots, z^{(n)}) \in \mathbb{R}^n$ and $r > 0$, let

$$Q(z,r) = \bigl\{x = \bigl(x^{(1)}, \ldots, x^{(n)}\bigr) \in \mathbb{R}^n \colon \bigl|x^{(i)} - z^{(i)}\bigr| < r/2,\ i = 1, \ldots, n\bigr\}.$$

We denote $cQ = Q(z, cr)$ for $Q = Q(z,r)$ and $c > 0$. For a measurable set $\Omega \subset \mathbb{R}^n$, we denote the characteristic function of Ω by χ_Ω.

Definition 4. For $1 \leqslant p < \infty$ and a function $\phi : (0, +\infty) \to (0, +\infty)$ with the doubling condition, let

$$\mathcal{L}_{p,\phi}\bigl(\mathbb{R}^n\bigr) = \bigl\{f \in L^p_{\mathrm{loc}}\bigl(\mathbb{R}^n\bigr) \colon \|f\|_{\mathcal{L}_{p,\phi}} < +\infty\bigr\},$$

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_{Q=Q(z,r)} \frac{1}{\phi(r)} \left(\frac{1}{|Q|} \int_Q \bigl|f(x) - f_Q\bigr|^p \, dx\right)^{1/p},$$

$$\text{where } f_Q = \frac{1}{|Q|} \int_Q f(x)\, dx.$$

If $\phi(r) = r^{(\lambda-n)/p}$ $(0 \leqslant \lambda \leqslant n+1)$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ which is the classical Campanato space. If ϕ is almost increasing, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ for all $p > 1$. We denote $\mathcal{L}_{1,\phi}(\mathbb{R}^n)$ by $\mathrm{BMO}_\phi(\mathbb{R}^n)$. If $\phi \equiv 1$, then $\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $\phi(r) = r^\alpha$, $0 < \alpha \leqslant 1$, then it is known that $\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

The following lemma can be proved by the method in [2, pp. 295–296].

Lemma 5. *Let Φ, q and U satisfy the assumptions of Theorem* 3. *Let $\phi(r) = 1/(r^n\Phi^{-1}(1/r^n))$ and $1/q + 1/q' = 1$. If $f \in L^{q,0}_{\mathrm{comp}}(\mathbb{R}^n)$ and $f = \sum_j \lambda_j a_j$ is any atomic decomposition in $H^{\Phi,q}_U(\mathbb{R}^n)$, then*

$$\int f\varphi = \sum_j \lambda_j \int a_j \varphi \quad \text{for } \varphi \in L^\infty_{\mathrm{comp}}\bigl(\mathbb{R}^n\bigr) \cap \mathcal{L}_{q',\phi}\bigl(\mathbb{R}^n\bigr).$$

Remark 6. Under the assumptions of the lemma, ϕ is almost increasing and $\phi(r)/r$ is almost decreasing. Assume that

$$\sup_{0<s<1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0).$$

Then we can show that the dual of $H^{\Phi,q}_U(\mathbb{R}^n)$ is $\mathrm{BMO}_\phi(\mathbb{R}^n)$ [4].

Remark 7. If φ is Lipschitz continuous and has a compact support, then $\varphi \in L^{\infty}_{\mathrm{comp}}(\mathbb{R}^n) \cap \mathcal{L}_{q',\phi}(\mathbb{R}^n)$.

4. Proof of the main theorem

Let $\phi(r) = 1/(r^n \Phi^{-1}(1/r^n))$. Let $f \in L^{q,0}_{\mathrm{comp}}(\mathbb{R}^n)$ and $\operatorname{supp} f \subset \overline{Q}$. We may assume that the center of the cube Q is the origin and that the sides are parallel to the axes. Let

$$Q = Q^0 = Q(0, R).$$

Since $L^{q,0}_{\mathrm{comp}}(\mathbb{R}^n) \subset H^{\Phi,q}_U(\mathbb{R}^n)$, f has an atomic decomposition in $H^{\Phi,q}_U(\mathbb{R}^n)$:

$$f = \sum_j \lambda_j b_j \quad \text{in } \mathfrak{D}' \tag{2}$$

with

$$\operatorname{supp} b_j \subset \overline{Q_j}, \quad Q_j = Q(z_j, r_j), \quad \int b_j = 0, \quad \|b_j\|_q \leqslant |Q_j|^{1/q} \Phi^{-1}\bigl(1/|Q_j|\bigr). \tag{3}$$

We make a new atom a_j from b_j and construct another decomposition for f:

$$f = \sum_j \lambda_j a_j \quad \text{in } \mathfrak{D}' \tag{4}$$

with

$$\operatorname{supp} a_j \subset \overline{Q^0}, \quad \int a_j(x)\,dx = 0, \tag{5}$$

and

$$\operatorname{supp} a_j \subset \overline{Q'_j}, \quad \|a_j\|_q \leqslant C\bigl|Q'_j\bigr|^{1/q} \Phi^{-1}\left(\frac{1}{|Q'_j|}\right), \tag{6}$$

where the constant $C > 0$ is dependent only on n and ϕ.

For $\nu = (\nu^{(1)}, \dots, \nu^{(n)}) \in \mathbb{Z}^n$, we define $\omega^\nu : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\omega^\nu(x) = \omega^\nu\bigl(x^{(1)}, \dots, x^{(n)}\bigr) = \bigl((-1)^{\nu^{(1)}} x^{(1)} + R\nu^{(1)}, \dots, (-1)^{\nu^{(n)}} x^{(n)} + R\nu^{(n)}\bigr).$$

Let $Q^\nu = Q(R\nu, R)$ and $\chi^\nu = \chi_{Q^\nu}$. Then $\omega^\nu(Q^0) = Q^\nu$. For the atomic decomposition (2) and (3), we make a_j from b_j as follows:

$$a_j = \sum_{\nu \in \mathbb{Z}^n} b^\nu_j, \qquad b^\nu_j = \bigl(b_j \chi^\nu\bigr) \circ \omega^\nu \quad \text{(see Fig. 1)}.$$

Since every b_j has a compact support,

$$\mathcal{N}_j = \bigl\{\nu \in \mathbb{Z}^n\colon \operatorname{supp} b_j \cap Q^\nu \neq \emptyset\bigr\}$$

is a finite set and $b^\nu_j \equiv 0$ for $\nu \notin \mathcal{N}_j$. It follows that $\operatorname{supp} b^\nu_j \subset \overline{Q^0}$ and $\int a_j = \int b_j$. Then we have (5). In the following we prove (6) and (4).

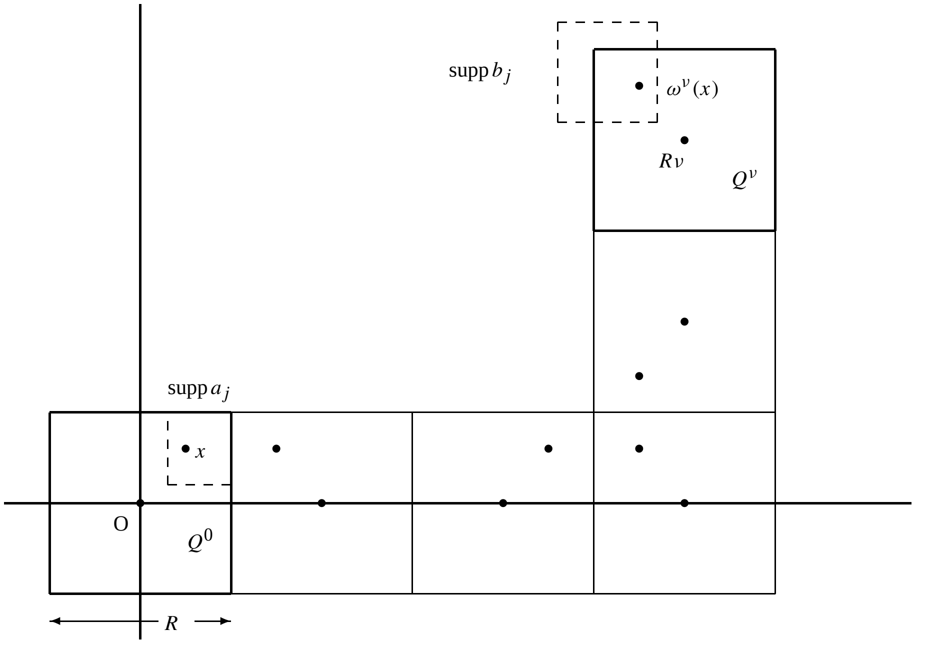

Fig. 1. Q^0, Q^ν and ω^ν.

Proof of (6). We use the almost decreasingness of $r\Phi^{-1}(1/r)$ and $r^{1/q}\Phi^{-1}(1/r)$.

If $r_j \leqslant R$, then $\mathcal{N}_j$ consists of 2^n elements at most and there exists a cube $Q_j' = Q(z_j{}', r_j{}')$ such that

$$\operatorname{supp} a_j \subset \overline{Q_j'} \subset \overline{Q^0} \quad \text{with } r_j/2 \leqslant r_j{}' \leqslant r_j.$$

Hence we have

$$\begin{aligned}\|a_j\|_q &\leqslant \sum_{\nu\in\mathcal{N}_j} \|b_j^\nu\|_q \leqslant 2^{n(1-1/q)}\|b_j\|_q \\ &\leqslant 2^{n(1-1/q)}|Q_j|^{1/q}\Phi^{-1}\big(1/\big|Q_j\big|\big) \leqslant C\big|Q_j'\big|^{1/q}\Phi^{-1}\big(1/\big|Q_j'\big|\big).\end{aligned}$$

If $r_j > R$, then $\mathcal{N}_j$ consists of $([r_j/R]+2)^n$ elements at most, where the notation $[s]$ represents the greatest integer less than or equal to the real number s, and $([r_j/R]+2)^n$ is comparable to $|Q_j|/|Q^0|$. Hence we have

$$\begin{aligned}\|a_j\|_q &\leqslant \sum_{\nu\in\mathcal{N}_j} \big\|b_j^\nu\big\|_q \leqslant \big([r_j/R]+2\big)^{n(1-1/q)}\|b_j\|_q \\ &\leqslant C\left(\frac{|Q_j|}{|Q^0|}\right)^{1-1/q}|Q_j|^{1/q}\Phi^{-1}\big(1/|Q_j|\big) \\ &= C\big|Q^0\big|^{1/q}\Phi^{-1}\big(1/\big|Q^0\big|\big)\frac{|Q_j|\Phi^{-1}(1/|Q_j|)}{|Q^0|\Phi^{-1}(1/|Q^0|)} \leqslant C\big|Q^0\big|^{1/q}\Phi^{-1}\big(1/\big|Q^0\big|\big).\end{aligned}$$

Proof of (4). We show that

$$\int f\varphi = \sum \lambda_j \int a_j\varphi \quad \text{for all } \varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n).$$

Let $\varphi = \varphi_1 + \varphi_2$, $\varphi_1, \varphi_2 \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, $\varphi = \varphi_1$ on Q^0, $\operatorname{supp}\varphi_1 \subset 2Q^0 = Q(0, 2R)$ and $\operatorname{supp}\varphi_2 \cap Q^0 = \emptyset$. Then

$$\int f\varphi_2 = \sum_j \lambda_j \int a_j\varphi_2 = 0.$$

So we may assume that $\operatorname{supp}\varphi \subset 2Q^0$. Let

$$\varphi^{-\nu} = (\varphi\chi^0) \circ (\omega^\nu)^{-1}, \qquad \chi^0 = \chi_{Q^0},$$

where $(\omega^\nu)^{-1}$ is the inverse function of ω^ν. Then $\operatorname{supp}\varphi^{-\nu} \subset \overline{Q^\nu}$ and

$$\begin{aligned}\int b_j^\nu\varphi &= \int b_j^\nu(\varphi\chi^0) = \int \left(b_j^\nu \circ (\omega^\nu)^{-1}\right)\left((\varphi\chi^0) \circ (\omega^\nu)^{-1}\right)\\ &= \int (b_j\chi^\nu)\varphi^{-\nu} = \int b_j\varphi^{-\nu}.\end{aligned}$$

We note that $\sum_{\nu\in\mathbb{Z}^n} \varphi^{-\nu}$ is Lipschitz continuous.

Let $A \geqslant 1$ with $\phi(s) \leqslant A\phi(r)$ and $\phi(r)/r \leqslant A\phi(s)/s$ for $r \geqslant s$. For all $\epsilon > 0$ there exist j_0, $K \in \mathbb{N}$ such that

$$\sum_{j>j_0} \lambda_j < \epsilon/C_{\varphi,R}, \qquad \bigcup_{j\leqslant j_0} \operatorname{supp} b_j \subset \bigcup_{j\leqslant j_0} \bigcup_{\nu\in\mathcal{N}_j} Q^\nu \subset KQ^0 = Q(0, KR),$$

where

$$C_{\varphi,R} = 2^n\left(2nA\|\varphi\|_\infty/\phi(R) + \|\varphi\|_{\mathcal{L}_{q',\phi}}\right).$$

Let $\theta \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and

$$0 \leqslant \theta(x) \leqslant 1, \qquad \left|\partial_i\theta(x)\right| \leqslant 2/R \quad (i = 1, \ldots, n), \qquad \theta(x) = \begin{cases} 1, & x \in KQ^0,\\ 0, & x \notin (K+1)Q^0,\end{cases}$$

and let

$$\psi = \theta\left(\sum_{\nu\in\mathbb{Z}^n} \varphi^{-\nu}\right) - \varphi, \qquad \psi_* = (1-\theta)\left(\sum_{\nu\in\mathbb{Z}^n} \varphi^{-\nu}\right).$$

Then ψ is Lipschitz continuous and $\operatorname{supp}\psi \subset \overline{(K+1)Q^0 \setminus Q^0}$. By Lemma 5 and Remark 7 we have

$$\sum_j \lambda_j \int b_j\varphi = \int f\varphi, \qquad \sum_j \lambda_j \int b_j\psi = \int f\psi = 0. \tag{7}$$

If $j \leqslant j_0$, then $\psi_* \equiv 0$ on $\bigcup_{\nu\in\mathcal{N}_j} Q^\nu$. If $j > j_0$, then there exists a constant c_j such that

$$\|\psi_* - c_j\|_{L^{q'}(Q_j)} \leqslant C_{\varphi,R}|Q_j|^{1/q'}\phi(r_j). \tag{8}$$

Actually, if $r_j > R$, then

$$\|\psi_*\|_{L^{q'}(Q_j)} \leqslant |Q_j|^{1/q'}\|\varphi\|_\infty \leqslant A|Q_j|^{1/q'}\|\varphi\|_\infty\phi(r_j)/\phi(R) \leqslant C_{\varphi,R}|Q_j|^{1/q'}\phi(r_j).$$

If $r_j \leqslant R$, then $\mathcal{N}_j$ consists of 2^n elements at most and $\psi_* = (1-\theta)\varphi^{-\nu}$ on Q^ν. For all $\nu \in \mathcal{N}_j$, there exists a cube $Q_j^\nu = Q(z_j^\nu, r_j)$ such that

$$Q_j \cap Q^\nu \subset Q_j^\nu \subset Q^\nu.$$

In this case we have

$$\int_{Q_j^\nu} \varphi^{-\nu} = \int_{Q_j^{\nu'}} \varphi^{-\nu'} \quad \text{for } \nu, \nu' \in \mathcal{N}_j.$$

Let

$$c_j = \bigl(1-\theta(z_j)\bigr)\frac{1}{|Q_j|}\int_{Q_j^\nu} \varphi^{-\nu} = \bigl(1-\theta(z_j)\bigr)\bigl(\varphi^{-\nu}\bigr)_{Q_j^\nu} \quad \text{for } \nu \in \mathcal{N}_j.$$

Then

$$\begin{aligned}
&\|\psi_* - c_j\|_{L^{q'}(Q_j)} \\
&\quad\leqslant \sum_{\nu\in\mathcal{N}_j} \bigl\|(1-\theta)\varphi^{-\nu} - c_j\bigr\|_{L^{q'}(Q_j^\nu)} \\
&\quad\leqslant \sum_{\nu\in\mathcal{N}_j} \Bigl(\bigl\|(1-\theta)\varphi^{-\nu} - \bigl(1-\theta(z_j)\bigr)\varphi^{-\nu}\bigr\|_{L^{q'}(Q_j^\nu)} + \bigl\|\bigl(1-\theta(z_j)\bigr)\varphi^{-\nu} - c_j\bigr\|_{L^{q'}(Q_j^\nu)}\Bigr) \\
&\quad\leqslant \sum_{\nu\in\mathcal{N}_j} \Bigl(\sup_{x\in Q_j^\nu}\bigl|\theta(x)-\theta(z_j)\bigr|\bigl\|\varphi^{-\nu}\bigr\|_{L^{q'}(Q_j^\nu)} + \bigl\|\varphi^{-\nu} - \bigl(\varphi^{-\nu}\bigr)_{Q_j^\nu}\bigr\|_{L^{q'}(Q_j^\nu)}\Bigr) \\
&\quad\leqslant 2^n\bigl(n(2/R)r_j|Q_j|^{1/q'}\|\varphi\|_\infty + |Q_j|^{1/q'}\phi(r_j)\|\varphi\|_{\mathcal{L}_{q',\phi}}\bigr) \\
&\quad\leqslant 2^n|Q_j|^{1/q'}\phi(r_j)\bigl(2nA\|\varphi\|_\infty/\phi(R) + \|\varphi\|_{\mathcal{L}_{q',\phi}}\bigr) = C_{\varphi,R}|Q_j|^{1/q'}\phi(r_j).
\end{aligned}$$

By (8) we have

$$\begin{aligned}
\left|\int b_j\psi_*\right| &= \left|\int b_j(\psi_* - c_j)\right| \leqslant \|b_j\|_q\|\psi_* - c_j\|_{L^{q'}(Q_j)} \\
&\leqslant |Q_j|^{1/q}\Phi^{-1}\left(\frac{1}{|Q_j|}\right)C_{\varphi,R}|Q_j|^{1/q'}\phi(r_j) = C_{\varphi,R} \quad \text{for } j > j_0.
\end{aligned} \tag{9}$$

We note that

$$\sum_{\nu\in\mathcal{N}_j}\varphi^{-\nu} = \begin{cases} \varphi+\psi, & j \leqslant j_0, \\ \varphi+\psi+\psi_*, & j > j_0, \end{cases} \quad \text{on } \operatorname{supp} b_j.$$

Hence

$$\begin{aligned}
&\sum_j \lambda_j \int a_j\varphi \\
&\quad= \sum_j \lambda_j \int \sum_{\nu\in\mathcal{N}_j} b_j^\nu\varphi = \sum_j \lambda_j \sum_{\nu\in\mathcal{N}_j}\int b_j^\nu\varphi = \sum_j \lambda_j \sum_{\nu\in\mathcal{N}_j}\int b_j\varphi^{-\nu} \\
&\quad= \sum_j \lambda_j \int b_j \sum_{\nu\in\mathcal{N}_j}\varphi^{-\nu} = \sum_{j\leqslant j_0}\lambda_j\int b_j(\varphi+\psi) + \sum_{j>j_0}\lambda_j\int b_j(\varphi+\psi+\psi_*)
\end{aligned}$$

$$= \sum_j \lambda_j \int b_j \varphi + \sum_j \lambda_j \int b_j \psi + \sum_{j>j_0} \lambda_j \int b_j \psi_*.$$

Using (7) and (9), we have

$$\left| \int f\varphi - \sum_j \lambda_j \int a_j \varphi \right| = \left| \sum_{j>j_0} \lambda_j \int b_j \psi_* \right| < \epsilon.$$

Therefore we have (4).

References

[1] D.-C. Chang, S.G. Krantz, E.M. Stein, H^p theory on a smooth domain in $\mathbb{R}^n$ and elliptic boundary value problems, J. Funct. Anal. 114 (1993) 286–347.
[2] J. García-Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland, Amsterdam, 1985.
[3] A. Miyachi, H^p spaces over open subsets of $\mathbb{R}^n$, Studia Math. 95 (1990) 205–228.
[4] E. Nakai, A generalization of Hardy spaces H^p by using atoms, preprint.
[5] K. Yabuta, A remark on the (H^1, L^1) boundedness, Bull. Fac. Sci. Ibaraki Univ. 25 (1993) 19–21.